TECHNIQUES FOR THERMAL CONDUCTIVITY MEASUREMENTS IN ANTARCTICA

by

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ABSTRACT

An accurate knowledge of the thermal properties of firn and ice within a glacier is essential for any reliable mathematical model of heat transfer. This paper considers the problem of determining the thermal properties of firn at Dome C, Antarctica, for use in such a model. First, the difficulties in accurately determining thermal properties are discussed. Then a physical experiment which can be performed under field conditions, but which will yield a well-posed mathematical problem for determining the unknown properties, is presented. Next, two different numerical techniques for solving the mathematical problem are discussed. Finally, some numerical approximations and error estimates are presented for the results of applying our numerical procedure to data from Dome C. Although insufficient data were obtained to test our methods fully, we have established a measurement procedure and a method of analysis which appear to be promising.

1. INTRODUCTION

We shall consider the problem of determining the thermal properties of firn at Dome C, Antarctica, for use in mathematical models for heat transfer in the glacier. Such models are very valuable for understanding the dynamics and stability of glaciers (Robin 1955, Bogoslovsky 1958, Jenssen and Radok 1961, 1963, Jenssen 1977, Weller and Schwerdtfeger 1977, Whillans in press), the effects of climate changes at the surface of the ice sheet, and the general process of heat transfer into the ice sheet. Accurate models could also be used to make reverse calculations from measured temperature profiles to derive past, decade-scale, climatic changes (Budd and others 1971, 1976, Johnsen 1975, Ewing in press [b], Ewing and Falk in press) and to guide the choice of locations, techniques and needed accuracies for future field measurements.

The basic model equation that we will consider will be

\[
\rho c_0 \frac{\partial \theta}{\partial t} = - \nabla \cdot (K \nabla \theta) + \nabla \cdot \mathbf{v} \theta + q,
\]

(1.1)

for the temperature \( \theta \) where \( \rho \) is the density, \( c \) is the specific heat, and \( K \) is the thermal conductivity of the firn or ice, and the subscript \( t \) denotes partial differentiation with respect to time. The \( \nabla \cdot \mathbf{v} \theta \) term models heat flow due to the physical transport of the firm. The term \( q \) is a measure of the heat generated internally. We shall concentrate on determining \( c \) and \( K \), assuming that the other properties are fairly well understood.

As a first step in our modeling process, we shall consider a one-dimensional model equation to describe the temperature distribution as a function of depth \( z \) into the glacier. Then, for \( 0 \leq z \leq D \), we consider

\[
\rho c_0 \frac{\partial \theta}{\partial t} = (K_0 \frac{\partial}{\partial z})^2 + \nabla \cdot \mathbf{v} \theta + q,
\]

(1.2)

where \( D \) is the thickness of the glacier. We make the physically motivated assumption that \( c \) and \( K \) change fairly slowly with depth, and we take small samples from firm cores in the field and determine the coefficients within each sample as a constant. We finally extrapolate the constants determined in this way to obtain spatially varying coefficients \( K \) and \( c \).

The thermal conductivity obtained through our measurement process includes heat conduction through the ice matrix, vapor transfer, sensible heat transfer by convection, and possibly radiative processes. The relative importance of these mechanisms is not now known and our work must be considered as obtaining an "effective" thermal conductivity for the firm.

Since our objective is to obtain error estimates for the coefficients determined, we developed a
2. DESCRIPTION OF THE PROBLEM

The measurement apparatus used by King (1979) is highly controlled laboratory setting which passes a large thermal gradient through a precisely measured sample of the material, allows the material to achieve thermal equilibrium, and then uses a steady-state temperature model to obtain the thermal properties. The process is usually repeated several times in order to obtain very precise values of the unknown properties.

The circumstances surrounding our measurement procedures are very different from those described above. The air and transport involving large temperature variations could dry out or melt the sample or radically change the thermal properties. Therefore the measurements must be made in the field instead of in a controlled laboratory environment. The "laboratory conditions" encountered at Dome C, Antarctica, were far from optimal. Temperatures of the "laboratory" were not within our control and varied diurnally, making it very difficult to obtain a static thermal equilibrium. Maintaining a constant voltage from the field batteries necessary for the heating and measuring process was also very difficult.

Next, a large temperature gradient placed across a thin firn sample would melt the sample and no further measurements could be made. Thus, a fairly low temperature gradient and a fairly thick sample are needed. Low thermal gradients are also required for conductivity measurements on saturated rocks at permafrost temperatures, as described in King (1979). The measurement apparatus used by King (1979) is similar to ours, but controlled laboratory conditions allowed thermal equilibrium to be attained and steady-state models to be used. Since we could not allow thermal equilibrium to be reached at Dome C, we required a transient model in our measurement process.

In our experiments with firn samples, there are no transport terms or internal heat generation; thus, under the assumption that K is a constant within the sample, (1.2) can be written in the form

\[ \frac{\partial T}{\partial t} = A_0 \frac{\partial}{\partial z^2} \left[ \frac{\partial T}{\partial z} \right], \quad x(0, T), \]


Thus our mathematical problem is not well-posed in the mathematical sense (Douglas and Jones 1962, Cannon 1964, Cannon and du Chateau 1973, Falk 1978, Ewing and Falk 1979, in press). As in the differential equations model, we shall then discuss some data obtained by one of us (JFB) at Dome C, Antarctica, and present the numerical approximations and error estimates obtained through our procedure using this data.

The determination of a unique A from (2.2) and (2.3) for arbitrary data \( g_1, g_2, f_0, \) and \( H \) is not possible. For example, if \( g_1, g_2 \) and \( f_0 \) are 0, then \( H = 0 \) and there is no heat flow. Clearly any \( A > 0 \) would then satisfy (2.2) and (2.3) with zero data. Thus our mathematical problem is not well-posed in the mathematical sense (Douglas and Jones 1962, Cannon 1964, Cannon and du Chateau 1973, Falk 1978, Ewing and Falk 1979, in press). We are thus faced with three major problems: (1) to find types of data \( f_0, g_1, g_2, H \), and assumptions on A which allow us to prove that a solution to (2.2) and (2.3) exists; (2) to set up an experiment which can be performed in the field which will yield the data needed in (1), and (3) to do a complete error analysis and interpretation of the resulting model problem.

3. DESCRIPTION OF THE MEASUREMENT APPARATUS AND PROCEDURE

We shall next describe the experimental apparatus and the experiment which was performed at Dome C, Antarctica, to yield our field data. The physical apparatus consists of a stack of control cylinders of lucite, the sample cylinder, and plates of copper containing thermistors. The stack is shown in Figure 1.

The apparatus was first tested in Antarctica during the 1978-79 field season. By allowing the stack to remain in operation for up to 12 h to approach

Fig. 1. Cross-section of apparatus used to measure thermal conductivity of firn samples. Apparatus is shown buried in a shallow pit.
were measured together with an estimated error toler­
ance at uniform intervals of 1 min for the first hour
and uniform intervals of 5 min for the duration of
the experiment. The distance of the
samples from the top of the core and their densities
ranged from 1 h 30 min to 2 h. The distance of the
stack was redesigned before the 1979-80 field
season by including a thick styrofoam sleeve for
better insulation around the stack. This reduced the
heat loss from the sides of the stack to a very low
level. Unfortunately, we were not able to run the
experiment to steady-state during the 1979-80 field
season to quantify this level closely.

The redesigned stack is well-insulated around the
top and sides, and was set onto the ice floor of the
measuring pit. The ice floor served as a heat sink,
the heater served as a heat source, and the ther­
mistors allowed the measurement of the time rate of
change of the temperature through the stack. The
thermistors allowed us to obtain temperature measure­
ments as a function of time on both sides of the
sample and on both sides of each lucite cylinder.
Since the thermal properties of the lucite were
known, we were able to solve initial boundary-value
problems in the lucite and thus determine the heat
flux at the ends of the sample and measure the total
heat flow through the stack, which indicated any
appreciable heat loss from the sides through the
insulation. The stack was allowed to reach an essen­
tially steady-state temperature distribution before
the heater was turned on. The initial temperature
was assumed to be linear through the sample and thus
determined by the initial temperature measurements
taken at the ends of the sample just prior to acti­
vating the heater. This thermal equilibrium also
yields the following compatibility conditions on the
data which are necessary for the numerical error
estimates:

\[ a) \quad g_1(0) = f_0(0), \quad g_2(0) = f_0(D'), \]
\[ b) \quad g_1(0) = g_1'(0) = g_2(0) = g_2'(0) = 0. \]  

In order to obtain as strong a gradient as possible
through the ice, and thus better error estimates as
described in Section 4, we wanted \( g_1(t) \) to rise
rapidly and \( g_2(t) \) to stay nearly constant or rise
slowly. This was the motivation for using the ice-pit
floor as a heat sink during the experiment.

The resistances of the thermistors (and thus the
temperatures via calibrations of the thermistors)
were measured together with an estimated error toler­
ance at uniform intervals of 1 min for the first hour
and uniform intervals of 5 min for the duration of
the experiment. The duration of the experiments
ranged from 1 h 30 min to 2 h. The distance of the
samples from the top of the core and their densities
were also carefully measured in the field. For a more
complete description of the measurement procedure
and equipment, see Ewing and others (1981).

4. THE MATHEMATICAL PROBLEM

In this section we shall present conditions which,
if satisfied, allow us to show that a solution to our
mathematical problem (2.2)-(2.3) exists, is unique,
and depends continuously upon the data. We shall then
give error estimates for the mathematical problem
based on these assumptions.

We normalize our problem. Let \( z = 0 \) and \( z = 1 \) be
the top and bottom of the sample, let \( g_1 = g_1(t) \) and
\( g_2 = g_2(t) \) be the measured temperatures at the top
and bottom of the sample, respectively, let \( H \) be the
measured heat flux at the top at some time \( t^* \in (0,1) \),
and let \( f_0 = f_0(z) \) be the initial linear temperature
distribution through the sample. We then obtain
(2.2, 2.3) with \( D' \) replaced by 1 and

\[ f_0(z) = (1 - z)g_z^1(0) + zg_2(0). \]

We shall seek \( A \) satisfying (2.2) and (2.3) and the
additional physical bounds

\[ 0 < A < A_1 < A^* \]  

From the field experiment, we see that \( g_1(t) \) in­
creases much faster than \( g_2(t) \). In particular, we have
from our experimental data that for \( t^* \) from (2.3)

\[ a) \quad g_1(t) > 4g_2'(t), \quad 0 < t < t^*, \]
and
\[ b) \quad g_1(0) > g_2(0). \]

Using Fourier series techniques, and techniques from
Carslaw and Jaeger (1959) and Cannon (1964), we can
obtain a solution, depending upon \( A_1 \), for (2.2) and
(2.3) of the form for \( 0 < t \) and \( 0 < t < T \).

\[ \phi(z, t; A) = (1 - z)g_1^1(0) + zg_2(0) \]  

where

\[ M(t, \sigma) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left( - (z - 2n)^2 \right) dz, \quad \sigma > 0. \]

Next, for \( \alpha \), we define the continuous function

\[ Q(\alpha) = - \alpha \phi(0, t^*; \alpha). \]

We must then find \( A_1 \) such that

\[ Q(A_1) = H. \]

In Ewing and others (1981), it was shown
that for \( \alpha \in [A_1, A^*] \), and \( Q \) and \( \delta \) defined above,
we have

\[ \frac{dQ(\alpha)}{d\alpha} \geq \rho(\phi_1(0) - \phi_2(0)) + \frac{1}{2} \frac{t^*}{t_0} \int_{0}^{t_0} \sqrt{H(\tau^* - \tau)} d\tau \]  

Thus for \( \alpha \in [A_1, A^*] \) and data satisfying (4.2),
Q is monotonically increasing and continuous. These
facts allow us to obtain an existence and uniqueness
theorem for our problem as in Ewing and others (1981)
for \( H(0), Q(0), Q(A^*) \) and data satisfying (4.2).

We next consider how this solution depends upon
measurement errors in the problem. Let

\[ \| f \| = \max_{t \in [0, 1]} |f(t)|. \]

Then assume \( H^*, \phi_1, \phi_1', \) and \( \phi_2 \) are obtained as \( H^* \), \( \phi_1^* \),
and \( \phi_2^* \) subject to measurement errors of the form

\[ a) \quad |H - H^*| \leq \epsilon_0, \quad \epsilon_0 > 0, \]
\[ b) \quad \| g_1 - g_1^* \| + \| g_2 - g_2^* \| \leq \epsilon_1, \quad \epsilon_1 > 0, \]
\[ c) \quad \| g_1' - g_1^* \| + \| g_2' - g_2^* \| \leq \epsilon_1', \quad \epsilon_1' > 0. \]
Also define $Q^*(a)$ as the analogue of $Q$ from (4.5) for the problem with $g_1$, $g_2$, and $H$ replaced by $g_1^*$, $g_2^*$, and $H^*$. Using the above notation, we obtain the following theorem which yields the continuous dependence of the solution of our problem upon the data.

**Theorem 4.1** If $a \in [A^*, A^*]$, $A$ satisfies (2.2) and (2.3), and $H^*$ satisfies

$$|Q^*(a) - H^*| \leq c_2,$$

then, for $G$ from (4.7) and $\varepsilon_0$, $\varepsilon_1$, and $\varepsilon_2$ from (4.9),

$$|a - A| \leq G^{-1}[(\varepsilon_0 + K_1 \varepsilon_1 + K_2 \varepsilon_2) + \varepsilon_2],$$

where $K_1 = 2\rho c A^*$ and $K_2 = \rho c[3(1 - t^* + t^*)]$.  

**Proof:** For details of the proof see Ewing and others (1981).

### 5. DESCRIPTION OF NUMERICAL METHODS

Before we discuss the method for numerically obtaining an approximate solution for (2.2) and (2.3), we describe how the numerical data were obtained. First the recorded resistance data from the thermistors were numerically converted to temperature data and smoothed very slightly, staying well within the error bars for the data (see Table I). This gave us $g_1^*$ and $g_2^*$ at 1 min intervals for the first hour with $\varepsilon_1$ from (4.9.b) approximately 0.03°C. Then, in order to obtain $H^*$ for (4.9.a), a separate boundary-value problem was solved numerically within the top lucite region where the thermal properties of lucite were known.

A piecewise linear Galerkin spatial discretization was used with a fourth order in time backward differentiation multistep method (Ewing in press [a], Bramble and Ewing in preparation*). A special startup procedure (Bramble and Ewing in preparation*) was required. The flux at the bottom of the lucite was then computed and used as $H^*$, the flux at the top of the sample, in Equation (4.9a). The numerical results are presented in Table I. Computer programs are available at reasonable expense.

From Theorem 4.1, if we can find a diffusivity $a^*(A^*, A^*)$ such that $Q^*(a^*)$ is close to our calculated flux $H^*$, then $a^*$ will be a good approximation to the unknown $A$. To determine such an $a^*$ computationally, first find $a_1$ and $a_2 \in [A^*, A^*]$ for which

$$Q^*(a_1) \leq H^* \leq Q^*(a_2).$$

Then pick sufficiently small error tolerance $\varepsilon_3$ and perform an interval-halving routine using $a_1$ and $a_2$ to start. At each step of the interval-halving routine, pick the mid-point $a$ of the active interval as a guess for $A$, numerically solve an initial-boundary-value problem using $a$, $g_1^*$, and $g_2^*$, and then compare the calculated flux using $a$ with $H^*$. The numerical procedures used in each step of this interval-halving routine are the fourth order multistep Galerkin procedures used for the lucite problem. The routine is terminated when $a_n$ is determined, satisfying

$$|Q^*(a_n) - H^*| \leq \varepsilon_3$$

where $Q^*(0, t^*; a_n)$ is the computed approximation of the derivative $\partial z / \partial t = 0$ and $t = t^*$ of the problem of determining $e(z, t; a)$ satisfying:

$$\begin{align*}
e_t &= a_0 e_{zz}, & & z \in (0,1), t \in (0, T], \\
e(0, t) &= g_1^*(t), & & t \in (0, T], \\
e(1, t) &= g_2^*(t), & & t \in (0, T], \\
e(z, 0) &= (1 - z) g_1^*(0) + z g_2^*(0), & & z \in [0,1].
\end{align*}$$

The numerical scheme used satisfies the estimate

$$c_4 = O((\Delta t)^4 + \Delta z),$$

$c_4$ can be made very small with the proper choice of the spatial mesh size $\Delta z$ and temporal step size $\Delta t$. Then, combining (5.2) and (5.4), we see that (4.10) is satisfied with $a^* = a_n$ and $c_2 = c_3 + c_4$. Using (4.11), we obtain

$$|a_n - A| \leq G^{-1}[(\varepsilon_0 + K_1 \varepsilon_1 + K_2 \varepsilon_1) + \varepsilon_3 + c_4],$$

an error bound for the accuracy in our coefficient determination problem. (5.6) is not a "sharp" estimate, but merely an upper bound for the error.

### TABLE I. NUMERICAL COMPUTATION OF $g_1$, $g_2$, AND $H^*$

<table>
<thead>
<tr>
<th>Run number</th>
<th>Depth (m)</th>
<th>Data smoothing error maximum norm</th>
<th>Data smoothing error mean square norm</th>
<th>$H^*$ (W m$^{-2}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$g_1(t)$</td>
<td>$g_2(t)$</td>
<td>$g_1(t)$</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0.0076</td>
<td>0.0048</td>
<td>0.0093</td>
</tr>
<tr>
<td>2</td>
<td>4.8</td>
<td>0.0011</td>
<td>0.0017</td>
<td>0.0009</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>0.0100</td>
<td>0.0055</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

*In preparation: J H Bramble and R E Ewing, "Efficient starting procedures for high order time-stepping methods for differential equations".
6. INTERSECTING GRAPH TECHNIQUES

So far, we have based our method for determining A (and thus K) upon the premise that we have a priori knowledge of the specific heat c. In many cases, we may not have accurate estimates of either the specific heat or the thermal conductivity. We next describe an intersecting graph technique used for similar problems, by Cannon and du Chateau (1973). We will obtain approximate pairs (K,c) for this more difficult problem.

Since we do not have a priori knowledge of c as before, we shall perform the same method with the flux measured at a fixed t = t* for a systematic sequence of values of c in the anticipated range. Each value of c will then determine a pair (c,K) through this procedure. If sufficiently many values of c are chosen at close intervals, we will in effect determine the "graph" of the "function" K = K(c) for the fixed time, t*. For another choice of t = t at some distance from t*, we can repeat the process with the same set of values for c to obtain a new "graph" of K = (c) for the same material. We hope that by taking radically different values of t we can obtain two curves with different properties. The true parameters (c,K) should lie on the intersection of the curves determined in this way.

We emphasize that the determination of each "point" (c,K) in this method entails the full numerical method of section 5. Therefore, since several "points" are necessary to determine two curves and their intersection by graphical techniques, this procedure requires considerably more computer time than the previous method. However, if one is uncertain of the values of c, one should always use this method of "check" the proposed values.

7. NUMERICAL RESULTS FROM DOME C DATA

In this section we shall present the numerical results obtained by applying our various methods to the data collected at Dome C, Antarctica. We then discuss data accuracies and corresponding error bounds. After presenting the results we shall compare them to previously known or assumed values of the various thermal properties under consideration and give our interpretation of the similarities and differences.

Only four samples were tested in our measurement apparatus at Dome C during the 1979-80 field season. On one of these test runs, the printer ran out of paper after twenty minutes and the run was aborted. The sample was later retested, but the data obtained were sufficiently anomalous that the results will not be presented. Thus the results of only three runs will be presented. The general data are given in Table II. The error in smoothing the data and the values of H* determined numerically are given in Table I.

As we have noted earlier, the basic numerical model described in section 5 required the specification of the specific heat of the sample. The temperature variations within all of the samples over the runs fell between the values of -29.9°C and -37.3°C. The first numerical results were obtained using an estimate for the specific heat of ice in this temperature range of 1.88 x 10^3 J kg^{-1} K^{-1}.

The numerical results obtained using this value for specific heat and t* = 60 min are given in Table III. To start the numerical procedures, we used A* = 1.7 x 10^{-7} and A* = 1.7 x 10^{-6} in units of m^2 s^{-1}. As the procedures ran, better choices of A* and A* were obtained.

<table>
<thead>
<tr>
<th>Sample number</th>
<th>Depth (m)</th>
<th>Thickness (m)</th>
<th>Density (kg m^{-3})</th>
<th>Date obtained</th>
<th>Date measured</th>
<th>Duration of run (min)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>0.748</td>
<td>395±14</td>
<td>Dec 17</td>
<td>Dec 21</td>
<td>110</td>
</tr>
<tr>
<td>2</td>
<td>4.8</td>
<td>0.748</td>
<td>370±13</td>
<td>Dec 21</td>
<td>Dec 23</td>
<td>120</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>0.748</td>
<td>426±16</td>
<td>Dec 28</td>
<td>Dec 31</td>
<td>120</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sample number</th>
<th>Depth (m)</th>
<th>Density (kg m^{-3})</th>
<th>Diffusivity (m^2 s^{-1})</th>
<th>Diffusivity (m^2 a^{-1})</th>
<th>Conductivity (W m^{-1} K^{-1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>395</td>
<td>1.05x10^{-6}</td>
<td>32.3</td>
<td>0.762</td>
</tr>
<tr>
<td>2</td>
<td>4.8</td>
<td>370</td>
<td>1.16x10^{-6}</td>
<td>35.7</td>
<td>0.790</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>428</td>
<td>1.00x10^{-6}</td>
<td>31.0</td>
<td>0.794</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Depth (m)</th>
<th>Density (kg m^{-3})</th>
<th>Specific heat (J kg^{-1} K^{-1})</th>
<th>Diffusivity (m^2 s^{-1})</th>
<th>Diffusivity (m^2 a^{-1})</th>
<th>Conductivity (W m^{-1} K^{-1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>395</td>
<td>2380</td>
<td>7.74x10^{-2}</td>
<td>23.9</td>
<td>0.683</td>
</tr>
<tr>
<td>4.8</td>
<td>370</td>
<td>2340</td>
<td>7.75x10^{-2}</td>
<td>24.0</td>
<td>0.665</td>
</tr>
<tr>
<td>8</td>
<td>428</td>
<td>2050</td>
<td>8.53x10^{-2}</td>
<td>26.4</td>
<td>0.734</td>
</tr>
</tbody>
</table>
Since we realized that the value of $1.88 \times 10^3$ for $c$ used in the numerical procedure was only an approximation based on the value for ice and since the specific heat depends to some extent upon density, we decided to use the intersecting graph technique described in section 6 on the same data to estimate both the specific heat and the thermal conductivity simultaneously. This procedure was carried out for each sample using $\Delta t = 60$ min and $\Delta z = 90$ min. The numerical results obtained by using the intersecting graph technique are presented in Table IV.

We note that the specific heats determined numerically from the field data were all somewhat higher than the specific heat of ice at the given temperature. The higher values of specific heat give lower values of diffusivity and conductivity. The values of thermal conductivity obtained here agree fairly well with the values obtained by Dalrymple and others (1966), which are interpreted by Weller and Schwerdtfeger (1977). We also note that the values of diffusivity in units of $m^2 s^{-1}$ presented in Table IV are very close to the value of $24.6 m^2 s^{-1}$ obtained using a slight linearization of the model presented by Lax (1979). The diffusivities obtained by using $c = 1.88 \times 10^3$ and presented in Table III, however, are much higher than the $24.6 m^2 s^{-1}$ estimate. We also note that conductivities correlate fairly well with density for this very narrow range of samples. If this correlation were understood better and were shown to hold in more widely varying circumstances, it might be used to help model diffusivity more accurately over wide variations of densities.

We point out that although the diffusivities obtained from Table IV are close to the expected values, the specific heats, and thus the thermal conductivities, are somewhat higher than expected. We also note that the thermal conductivity of ice is in the range 2.1 to $0.92 W m^{-1} K^{-1}$, and we would expect the conductivity of firn to be somewhat lower.

Next, we briefly discuss error bounds for our methods. We emphasize again that the estimates obtained in (5.6) are not sharp, but are merely obtained from Table IV are close to the expected value problem we can argue as Ewing and others (1981). Thus, combining (7.1)-(7.4), we obtain the estimate

$$|a_n - A| < \delta^2(0.22).$$

(7.6)

We can then obtain an approximate error tolerance from an estimate on the size of $\delta$. We shall present an estimate of $G$ for the 4.8 m sample. Estimates for the other samples are obtained in an analogous fashion. For this sample

$$g_1(0) = g_2(0) = 1.18$$

(7.7)

and

$$60 \int_0^{1/\sqrt{4}(90-t)} \frac{g_1(t) - 4g_2(t)}{\sqrt{A(t)}} dt = 0.553.$$ 

(7.8)

Thus, from (4.7) we see that

$$G^{-1} = 6.0 \times 10^{-6}.$$ 

(7.9)

Then, combining (7.6) and (7.9), we obtain the bound on the error tolerance in $m^2 s^{-1}$ of

$$|a_n - A| \leq 1.3 \times 10^{-6}.$$ 

(7.10)

We emphasize that this is an upper bound for the error since (7.9) is a gross upper bound on $(\frac{\partial U}{\partial a}(\bar{a}))^{-1}$.

Usually the $K_{E1}$ term in (4.11) will dominate, as with our data. If we assume $K_{E1} \ll K_{E2}$ as an "estimate" of our error, we see that (7.5) can be replaced by

$$\varepsilon_0 = 0.025,$$ 

(7.11)

and (7.7) can be replaced by

$$|a_n - A| \leq \left( \frac{\partial U}{\partial a}(\bar{a}) \right)^{-1} (0.05).$$ 

(7.12)

For $c = 2440$ in the 4.8 m sample we use the results of the interval halving scheme to obtain the estimate

$$\frac{\partial U}{\partial a}(0.43) = 3.6 \times 10^5.$$ 

(7.13)

Combining (7.12) and (7.13) we obtain the estimate in $m^2 s^{-1}$

$$|a_n - A| \leq 1.3 \times 10^{-7}.$$ 

(7.14)

Our accuracy is basically limited by the data measurement accuracy and not by the mathematical and computational tools used. We also found that, although the numerical methods were somewhat complex, the results indicated the stability of the methods by producing very smooth "curves" in the intersecting graph technique.

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REFERENCES

Bogoslovskiy V N 1958 The temperature conditions (regime) and movement of the Antarctic glacier shield. International Association of Scientific Hydrology Publication 47 (Symposium of Chamonix – Physics of the Motion of Ice): 287-305

Budd W F, Jenssen D, Radok U 1971 Derived physical characteristics of the Antarctic ice sheet, Mark 1. Melbourne, University of Melbourne. Meteorology Department (Publication 18)

Budd W F, Young N W, Austin C R 1976 Measured and computed temperature distributions in the Law Dome ice cap, Antarctica. Journal of Glaciology 16(74): 310


Ewing R E In press[b]. Shallow-depth temperature models for Dome C, Antarctica. Antarctic Journal of the United States


Johnsen S J 1977 Stable isotope profiles compared with temperature profiles in firm with historical temperature records. International Association of Hydrological Sciences Publication 118 (General Assembly of Grenoble 1976 – Isotope and Impurities in Snow and Ice): 388-392


Lax P 1971 Conductive and convective energy transfer processes in polar firm. Ohio State University. Institute of Polar Studies. Report 72

Robin G de Q 1955 Ice movement and temperature distribution in glaciers and ice sheets. Journal of Glaciology 2(10): 503-532


Whillans I M In press. Inland ice sheet thinning due to Holocene warmth. Science